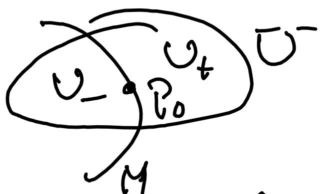
Lecture 18

Local holom. extension of CR fcns.

Let $\rho = 0$ be a local def. eq. for $M \subseteq \mathbb{C}^{n+1}$ near $p_0 \in M$. Let $U \subseteq \mathbb{C}^{n+1}$ be small open nbhd of p_0 s.t. $U \setminus M$ has two connected comp.

$U \setminus M = \{\rho < 0\} \cup \{\rho > 0\} = U_- \cup U_+$



Let $\theta = i\partial\rho|_{\mathbb{C}TM} = \iota^*(i\partial\rho)$ $(\iota : M \hookrightarrow \mathbb{C}^{n+1})$

<u>Thm</u> (Lewy Extension Thm). Suppose the Levi form $\mathcal{L}^{\theta}_{p_0}$ of M at p_0 has a positive eigenvalue. Then, $\exists\, p_0 \in V \overset{\text{open}}{\subseteq} U$ s.t. every C^k CR function in $M \cap U$ extends holomorphically to $V_- = \{z \in V : \rho(z) < 0\}$.

For the pf, we need the following:

<u>Prop</u> Given $M \subseteq \mathbb{C}^{n+1}$, $p_0 \in M$, and U s.t. $U \setminus M = U_- \cup U_+$ as above. Suppose the Levi form $\mathcal{L}^{\theta}_{p_0}$ has l_+, l_-, r pos, neg, zero eigenvalues, resp. Then, $\exists$ local coord's $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ s.t. $p_0 = (0,0)$ and a defining function ρ in $0 \in V \subseteq U$ s.t. $\rho < 0$ in $V_- = V \cap U_-$ and

$$\rho = -\operatorname{Im} w + \varphi(z, \operatorname{Re} w), \text{ where}$$

$$\varphi(z,s) = \sum_{1}^{l_+} |z_j|^2 - \sum_{l_+ + 1}^{l_+ + l_-} |z_j|^2 + O(|(z,s)|^3).$$

<u>Pf.</u> Let $\tilde{\rho}$ be any def. fcn. s.t. $\tilde{\rho} < 0$ in U_- and $\tilde{\rho} > 0$ in U_+. Note $\xi = \left(\frac{\partial \tilde{\rho}}{\partial z_1}(p_0), \ldots, \frac{\partial \tilde{\rho}}{\partial z_{n+1}}(p_0)\right) \neq 0$.

After a translation and cplx linear transformation, WLOG: $\xi = (0, \ldots, 0, -i) \Rightarrow$

$$\tilde{\rho}(\tilde{z}) = -\operatorname{Im} \tilde{z}_{n+1} + \sum_{i,j=1}^{n+1} a_{ij} \tilde{z}^i \bar{\tilde{z}}^j + \operatorname{Re} \sum_{i,j=1}^{n+1} b_{ij} \tilde{z}^i \tilde{z}^j + O(|\tilde{z}|^3)$$

Let's make the change of variables

$\tilde{z}_j \to \tilde{z}_j$ for $j=1,\dots,n$, and $\tilde{z}_{n+1} \to \tilde{z}_{n+1} + i\sum_{i,j=1}^{n+1} b_{ij}\tilde{z}_i\tilde{z}_j$

Then, in the new coords,

$$\tilde{\rho} = -\mathrm{Im}\,\tilde{z}_{n+1} + \sum_{i,j=1}^{n+1} a_{ij}\tilde{z}_i\bar{\tilde{z}}_j + O(|z|^3)$$

In order to write ρ as in the Prop, we must use the implicit fcn thm to solve for $\mathrm{Im}\,\tilde{z}_{n+1}$, but before doing so, we must make one more transformation. Note that $|\mathrm{Im}\,z|^2 = \frac{1}{4}|z-\bar{z}|^2 = \frac{1}{4}(z^2+\bar{z}^2-2|z|^2)$

We now introduce the coordinates (z,w):

$$\tilde{z}_j = z_j, \quad j=1,\dots,n; \quad \tilde{z}_{n+1} = w + 2i\sum_{i=1}^{n} a_{i,n+1} z_i w - \frac{1}{2}a_{n+1,n+1}w^2.$$

Plugging into $\tilde{\rho}$ yields

$$\tilde{\rho} = -\mathrm{Im}\,w + \sum_{i,j=1}^{n} a_{ij}z_i\bar{z}_j + \sum_{i=1}^{n} a_{i,n+1}z_i(\bar{w}-w) + \sum_{i=1}^{n} a_{n+1,i}\bar{z}_i(w-\bar{w}) - 2a_{n+1,n+1}|\mathrm{Im}\,w|^2 + O(|(z,w)|^3),$$

We note that the linear term in $\tilde{\rho}$ is $-\operatorname{Im} w$ and the quadratic terms only depend on z and $\operatorname{Im} w$ $\left(= \frac{1}{2i}(w - \bar{w})\right)$.
If we use the implicit function thm to solve for $\operatorname{Im} w$ in $\tilde{\rho} = 0$, then we find $\operatorname{Im} w = \varphi(z, \operatorname{Re} z)$, where

$$\varphi(z, s) = \sum_{i,j=1}^{n} a_{i\bar{j}} z_i \bar{z}_j + O(|(z,s)|^2)$$

By making a final cplx linear transf. of the z, we can diagonalize and rescale eigenvalues of $(a_{i\bar{j}})_{i,j=1}^{n}$ to having only ± 1 and 0 on the diagonal. The statement in Prop. follows. $\boxtimes$

Pf of Lewy's Ext. Thm.

In the normalization of Prop,

$$\rho(z_1, 0, \dots, 0) = |z_1|^2 + O(|z_1|^3) \Rightarrow$$

$\exists\, \delta > 0$ s.t. $\rho(z_1, 0, \dots) > 0$ for $0 < |z_1| \leq \delta$.

By cont., we can find $\varepsilon > 0$ s.t.

$$\rho(z, w) > 0,$$

for $\frac{\delta}{2} \leq |z_1| \leq \delta$ and $\underbrace{|z_2|^2 + \dots |z_n|^2 + |w|^2 \leq \varepsilon^2}_{|(z', w)|^2 \leq \varepsilon^2}$.

We let $V = \{|z_1| < \delta,\ |(z', w)| < \varepsilon\} = \underset{\rho < 0}{V_-} \cup M \cup \underset{\rho > 0}{V_+}$.

What does V_- look like? Fix $|(z', w)| < \varepsilon$ and let $\mu(z_1) = \rho(z_1, z'; w)$. The function μ has the form

$$\mu(\zeta) = -r + \mathrm{Re}(\alpha\zeta) + \mathrm{Re}(\beta\zeta^2) + (1-\gamma)|\zeta|^2 + O(|\zeta|^3),$$

Where r, α, β, γ are small (bdd by some constant times ε).

Note that $\Delta\mu = 4\frac{\partial^2}{\partial z\partial\bar{z}}\mu = 1-\gamma + O(|z|)$.

Thus if we make ε, δ suff. small, then μ is subharmonic in $|z|<\delta$. $\Rightarrow$

$\{\mu<0\}$ is simply connected or empty.

Why? Recall $\mu>0$ on $\frac{\delta}{2}\leq|z|<\delta$. If $\{\mu<0\}\neq\emptyset$ and not s.c. then $\{\mu\geq 0\}$ would have a nonempty component that is compact in $|z|<\delta/2$ (since it is disjoint from the component containing $\frac{\delta}{2}\leq|z|<\delta$).

But then μ takes a local max in $\{\mu\geq 0\}$.

This cannot happen by Max Mod Princ., since μ is SH.

Next, we note that if $(z', w) = (0, r)$ then $\mu(z) = -r + |z|^2 + O(|z|^3) \Rightarrow$ $\{\mu<0\}\neq\emptyset$ (contains a small disk centered at 0) if $r>0$ but $\{\mu<0\}=\emptyset$ if $r<0$ (if δ is small enough).

From here we can finish the proof in 2 alt. ways.

(1) Baouendi–Treves Approximation Thm.

Let $M \subseteq \mathbb{C}^N$ be a CR mfld and $p_0 \in M$. Then, $\exists W \overset{\text{open}}{\subseteq} M$ s.t. $p_0 \in W$ and any C^k CR function u can be approximated in C^k-norm on W by holom. polynomials $P(z)$.

For pf. see [BER], §2.4.

With BT approximation, the LET follows easily by the Max Mod Principle. Let δ, ε be so small that $V \cap M \subseteq W$. Let $P_\nu(z)$ be a seq. of polynomials $\to u$ in C^k on W. Clearly, P_ν conv. unif on the polynomial hull of W.

<u>Claim</u> $V_- := \{z \in V : \rho(z) < 0\} \subseteq$ polynomial hull of W.

Pf. Let $(z, w) \in V_-$. Then $|(z', w)| < \varepsilon$ and (in the notation above), $z_1 \in \{\mu < 0\} \Rightarrow$ $\{\mu < 0\} \neq \emptyset$ and, hence, $\{\mu < 0\}$ is an open simply connected subset of $|\zeta| < \frac{\delta}{2}$.

$$\Rightarrow |P(z, z', w)| \leq \sup_{\zeta \in \{\mu = 0\}} |P(\zeta, z', w')| \leq \sup_{\overline{W}} |P|. \quad \boxtimes$$

Since P_ν conv. unif. on V_-, their limit is a holom. function $v \in \mathcal{O}(V_-)$ and $v|_M = u$.

This completes the proof of LEV. $\boxtimes$

An alternative approach, not using BT appr.

(2) As in pf. of Hartogs Thm–CR version, we can extend u to a $\mathcal{C}^k$ function u_1 in V_- s.t. $u_1|_M = u$. We can improve u_1 to a $\mathcal{C}^k$ function u_2 in V_- s.t. $\bar\partial u_2 = O(\rho^2)$ and $u_2|_M = u$. (see proof of Hartogs–CR in Lecture notes 17).

We can then proceed as in pf of Hartogs Ch.
Let $f = -\bar{\partial} u_2$ in V_-, $f = 0$ in $V_+ \supseteq \{\frac{\delta}{2} \le |z_1| < \delta\}$. Then, $f \in C^1$ in V and $\operatorname{supp} f \subseteq \{|z_1| < \frac{\delta}{2}\}$. If we define $v(z) = \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \dots, w)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}$ as before, then $\bar{\partial} v = f$ (same pf since $f \in C^1$), v is holom. in V_+, and $v = 0$ when $|(z', w)| < \varepsilon$ is s.t. $\{\mu < 0\} = \emptyset$ (e.g. for $w = s + it$, $|(z', s)| << \varepsilon$ and $t < 0$).
Moreover, since for each $|(z', w)| < \varepsilon$, $\{\mu < 0\}$ is simply connected (or $\emptyset$), it is easy to see that V_+ is connected.
Thus, by uniqueness, $v = 0$ in V_+.
Then, $u_0 = u_2 + v$ is holom. in V_- and $u_0|_M = u$ (as in pf of Hartogs). ∎

Rem. (1) In Alt.(2), we extended u to u_2 s.t. $\bar\partial u_2 = O(\rho^2) \Rightarrow f = -\bar\partial u_2$ in V_-, $f=0$ on $\bar V_+$ is $\mathcal{C}^1$. We could continue the improvements of u_1 to u_k s.t. $\bar\partial u_k = O(\rho^k) \Rightarrow f \in \mathcal{C}^{k-1}$. This gives us as much smoothness of the holom. extension as u initially has.

Cor. Let $M \subseteq \mathbb{C}^{n+1}$, $p_0 \in M$, and U as in L.E.T. Suppose Levi form $\mathcal{L}^\theta_{p_0}$ has a positive and a negative eigenvalue, then $\exists\, p_0 \in V \underset{\text{open}}{\subseteq} U$ s.t. for every CR function u in $M \cap U$ $\exists$ holom. v in V s.t. $v|_M = u$.

Pf. L.E.T gives v_+ in V_+ and v_- in V_-, holom. in their domain s.t. both extend $\mathcal{C}^k$ to M with $v_\pm|_M = u$. Thus, we get a $\mathcal{C}^k$ function u in V s.t. u is holom. in $V_+ \cup V_-$.

If $k \geq 1$, then clearly $\bar\partial u = 0$ also on M by conf. $\Rightarrow$ u is holom. in V. (This also true if $k=0$; Riemann removable sing. thm.). $\square$

Optimal conditions for holom. extension

<u>Recall</u>. $M \subseteq \mathbb{C}^{n+1}$ CR mfld is <u>minimal</u> at $p_0 \in M$ if $\nexists\, M' \subseteq M$ s.t. $p_0 \in M'$, $\dim M' < \dim M$, and $T^{1,0}_p M = T^{1,0}_p M'$, $\forall p \in M'$. When M is a real hypersurface, existence of such M' can (for dimensional reasons) only occur when $T^{1,0}M' \oplus T^{0,1}M' = \mathbb{C}TM'$. By Newlander-Nirenberg, M' is a complex mfld of $\dim_{\mathbb{C}} = n-1$ (or a cplx hypersurface/divisor).

<u>Thm</u> Let $M \subseteq \mathbb{C}^{n+1}$ be minimal at $p_0 \in M$. Then, $\exists\, p_0 \in U \overset{\text{open}}{\subseteq} \mathbb{C}^{n+1}$ s.t. $U \setminus M = U_0 \cup U_1$ (2 components) and for all CR functions u on M, $\exists\, v \in \mathcal{O}(U_0) \cap C^k(\overline{U_0})$ s.t. $v|_M = u$.

Rem. ① Thm due to Trepreau (1980). For CR mflds $M \subseteq \mathbb{C}^N$ of higher codimension (*), due to Tumanov (1990). The sufficiency of finite type ($\Rightarrow$ minimality) is due to Baouendi–Rothschild (UCSD) in the '80s.
② Minimality is also necessary (due to BR)

Thm. If all CR functions extend to one side of real hypersurface $M \subseteq \mathbb{C}^{n+1}$ near $p_0 \in M$, then M is minimal at p_0.

Pf. We shall prove it for real-analytic M, the pf in the C^∞ being a bit more involved. So, let's assume M is not minimal at $p_0 \Rightarrow \exists$ cplx hypersurface $p_0 \in \Sigma \subseteq M \Rightarrow$ we can find coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ s.t. $p_0 = (0,0)$ and $\Sigma = \{w = 0\}$. In the real-analytic case we can arrange it so that

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(*) In this case, there are no "sides". Extension is to "wedges".

M is given by the equation (recall $\{w=0\}\subseteq M$)

$$\operatorname{Im} w = (\operatorname{Re} w)\,\phi(z, \operatorname{Re} w)$$

s.t. $\phi(0,s)=0$.

(For more on general "normal" coordinates, see [BER] §4.2)

Note that on M then, $w = s(1+i\phi(z,s))$

$\Rightarrow \operatorname{Re} w^2 = s^2(1-\phi(z,s)^2) \geq \frac{1}{2}s^2$ for $|(z,s)|$ small. Of course, $h = w^2|_M$ is a smooth CR function on M, which then takes values in the right half plane (RHP). Thus, if we let $\zeta^{2/3}$ be the $\frac{2}{3}$-root defined in RHP, then $h^{2/3}$ is a CR function (by chain rule) on M.

<u>Claim</u> $h^{2/3}$ does not extend to either side of M.

<u>Pf of Claim</u>. Note that the w-plane $P=\{z=0\}\cong\mathbb{C}^1\subseteq\mathbb{C}^{n+1}$ intersects both

sides of M, with $M \cap P = \{\operatorname{Im} w = 0\}$ and the intersection with the 2 sides correspond to upper and lower half planes (UHP, LHP). If $h^{2/3}$ extended to either side the restriction to P would extend to either UHP or LHP. To show this cannot happen, we note that in the coordinates $(z,s) \to (z, s+is\phi)$ on M, $h^{2/3} = \left[(s+is\phi)^2\right]^{2/3} =$

$$= \left[s^2(1-\phi^2) + 2is^2\phi^2\right]^{2/3} \Rightarrow \{\phi(0,s)=0\}$$

$$h^{2/3}(0,s) = s^{4/3} \geq 0.$$

Let's assume $h^{2/3}(0,s)$ extends holom. to UHP (LHP being similar). Call the extension $g(w)$. Clearly in the quadrant $\text{UHP} \cap \{\operatorname{Re} w > \}$, $g(w) = w^{4/3}$ where we use the $\frac{4}{3}$-root corresponding to the principal branch of Log in $\mathbb{C} \setminus \{s<0\}$.

Thus, in the UHP,

$$g(w) = e^{\frac{4}{3}\operatorname{Log} w} = e^{\frac{4}{3}\log|w|}\, e^{\frac{4i}{3}\operatorname{Arg} w}$$

w/ $\operatorname{Arg} w \in [-\pi, \pi)$. But then, for $s<0$, $g(s) = |s|^{4/3} e^{\frac{4\pi}{3} i}$, which is not real, contradicting $g(s) = h^{2/3}(0,s) = s^{4/3}$. This proves the claim. ☒

Thus, we have exhibited a CR function that does not extend. Of course, $h^{2/3}$ is only C^1 on M, but if you want a C^∞ function, take $e^{-1/h^{2/3}}$. ☒